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# Infinite families of links with trivial Jones polynomial

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**Abstract**

For each $k \geqslant 2$, we exhibit infinite families of prime $k$-component links with Jones polynomial equal to that of the $k$-component unlink. 

## 1. Introduction

The startling discovery by Jones some 17 years ago of a polynomial invariant of links arising from von Neumann algebras [3] opened an entirely new vista in 3-dimensional topology. The Jones polynomial and its generalizations have been used to settle century-old conjectures in knot theory [4,6,7,12], and have led to new connections between topology and physics [13]. Despite these advances, it cannot yet be said that the Jones polynomial is well understood in terms of intrinsic topological properties of links; for example, at this writing it is unknown whether there exists a non-trivial knot indistinguishable by the Jones polynomial from the unknot.

In this article we produce a strong affirmative answer to the analogous question for links, in that we exhibit infinite families of prime $k$-component links with Jones polynomials equal to those of the corresponding unlinks, for all $k \geqslant 2$.

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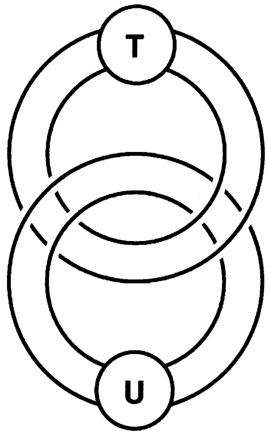



Fig. 1. The diagram $H(T,U)$.

We have not found any non-trivial links with trivial Homfly or Kauffman 2-variable polynomials; however, in Section 4 below we give an infinite sequence of prime 2-component links which neither the Jones nor the Alexander polynomial can distinguish from the unlink.

The links in these families are all satellites of the Hopf link, and all conform to the pattern $H(T,U)$ illustrated in Fig. 1, formed by clasping together the numerators of tangles $T,U$. Our method is based on a transformation $H(T,U) \to H(T,U)^{\omega}$, whereby the tangles $T,U$ are cut out and reglued by certain specific homeomorphisms of the tangle boundaries. Like mutation, the transformation $\omega$ preserves the Kauffman bracket polynomial; however, it is more effective in generating examples, as a trivial link can be transformed to a prime link, and repeated application yields an infinite sequence of inequivalent links.

Throughout this paper we shall work with the Kauffman bracket version of the Jones polynomial. We recall that the Kauffman bracket polynomial $\langle D\rangle \in \mathbb{Z}\,[a,a^{-1}]$ of a link diagram $D$ is defined by the following two properties:

(i) the bracket polynomial of a diagram consisting of $k$ disjoint simple closed curves in the plane is $\delta^{k-1}$, where $\delta = -a^{-2} - a^{2}$;

(ii) $\langle \times \rangle = a\langle )( \rangle + a^{-1}\langle \asymp \rangle$, where the three vignettes indicate diagrams that are identical except that a crossing of the first diagram is nullified in two different ways to form the second and third diagrams.

We also recall that if $D$ is a diagram of a link $L$, then the Jones polynomial $V_L(t)$ of $L$ is obtained by substituting $t = a^{-4}$ in the polynomial $(-a^3)^{-wr(D)}\langle D\rangle$, where $wr(D)$ is the writhe of the diagram $D$. Thus, the Jones polynomial of a $k$-component link represented by a diagram $D$ is trivial if and only if the bracket polynomial of $D$ is equal to $\delta^{k-1}(-a^3)^{wr(D)}$.

Historically [1], the symbol $-T$ denotes the reflection of $T$ in the projection plane, and $n$ denotes a twist of $n$ crossings proceeding from west to east, where the twist is right-handed if $n > 0$ and left-handed if $n < 0$ (some authors use the opposite convention). We note that the "integer" tangle $-n$ is indeed an additive inverse of the tangle $n$, in that their tangle sum is $0 = \asymp$; however, it is false in general that $-T$ is an additive inverse of $T$. We shall denote by $T^{\rho}$ the reflection of $T$ in a NW–SE axis, and by $T \cdot U$ the tangle sum $T^{\rho} + U$. Note that $T \cdot 0 = T^{\rho}$. Traditionally, for

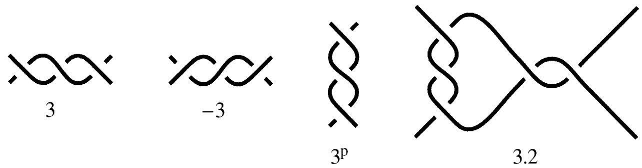



Fig. 2. Examples of tangles.

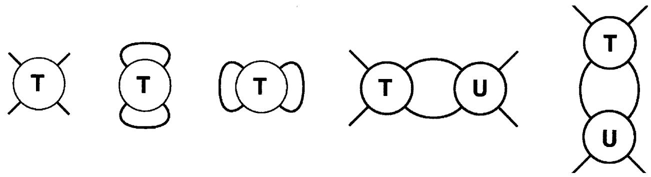



Fig. 3. A tangle $T$, its closures $T^{\mathrm{N}}, T^{\mathrm{D}}$ and the tangle combinations $T + U$, $T * U$.

$n \in \mathbb{Z}$, $n \neq 0$ the tangle $n^{\rho}$ is denoted $1/n$, and $0^{\rho} = )($ is denoted $\infty$. Regarding iterated "products", we follow the standard convention that $T \cdot U \cdot V$ is to be interpreted as $(T \cdot U) \cdot V$. Some examples are illustrated in Fig. 2.

Given a tangle $T$, we shall denote by $T^{\mathrm{N}}, T^{\mathrm{D}}$ the numerator and denominator closures of $T$, and by $T + U$ the tangle sum of $T, U$. We shall also have occasion to consider the "vertical sum" of two tangles $T * U = (T^{\rho} + U^{\rho})^{\rho}$ (Fig. 3).

We acknowledge with pleasure the computational knot theory package *K2K* by Ochiai and Imafuji [8] which was an essential tool for our investigations, and Stephenson's circle packing software *Circlepack* [11], whose underlying engine was used for generating the pictures of links.

## 2. Elementary algebraic and geometric properties of $H(T, U)$

Our immediate task is to derive a formula for the bracket polynomial of the link diagram $H(T, U)$ depicted in Fig. 1. We shall present two methods of obtaining such a formula. Our first approach is to apply the 2-strand parallel bracket expansion formula given in [5, Proposition 5, p. 33]. In that proposition, $[K]$ denotes $\langle K^2 \rangle$, where $K^2$ is the 2-strand parallel of $K$; also, the symbol denotes $\langle\ \rangle$. From Proposition 5(i) of [5] we immediately obtain the following "switching formula":

$$[\ ] - [\ ] = (a^4 - a^{-4})([\asymp] - [)(]) + (a^2 - a^{-2})([\ ] + [\ ] - [\ ] - [\ ]).$$

We apply this formula to either of the two "2-strand parallel" crossings of the clasp of $H(T, U)$. Switching this generalized crossing yields a diagram regularly isotopic to a split union of the numerators of $T, U$; therefore, its bracket polynomial is $\delta \langle T^{\mathrm{N}} \rangle \langle U^{\mathrm{N}} \rangle$. Each of the last four diagrams in the switching formula is regularly isotopic to a connected sum of the denominators of $T, U$; therefore, these terms cancel out.

The diagram corresponding to the term $[\asymp]$ is regularly isotopic to the numerator of the tangle sum $T + U$, modified by the insertion of the 2-strand parallel of a positive kink in the two strands issuing from the western (or eastern) ends of $T$; we therefore have, by means of a simple bracket calculation

$$[\asymp] = a^6\{a^2\langle (T+U)^{\mathrm{N}}\rangle + 2\langle T^{\mathrm{D}}\rangle\langle U^{\mathrm{D}}\rangle + a^{-2}\delta\langle T^{\mathrm{D}}\rangle\langle U^{\mathrm{D}}\rangle\}.$$

Similarly, we have

$$[\rangle\langle] = a^{-6}\{a^{-2}\langle (T+U)^{\mathrm{N}}\rangle + 2\langle T^{\mathrm{D}}\rangle\langle U^{\mathrm{D}}\rangle + a^{2}\delta\langle T^{\mathrm{D}}\rangle\langle U^{\mathrm{D}}\rangle\}.$$

After collecting terms, the switching formula yields

**Proposition 2.1.** $\langle H(T,U)\rangle = \delta\{\delta g\langle (T+U)^{\mathrm{N}}\rangle + \langle T^{\mathrm{N}}\rangle\langle U^{\mathrm{N}}\rangle - g\langle T^{\mathrm{D}}\rangle\langle U^{\mathrm{D}}\rangle\}$, *where* $g = a^{-8} - 2a^{-4} + 2 - 2a^4 + a^8$.

Following ideas developed in [10], we now introduce a formalism which will be useful in the next section, and which yields an alternative formula for $\langle H(T,U)\rangle$.

Given a tangle $T$, the bracket expansion formula $\langle \times \rangle = a\langle \rangle\langle \rangle + a^{-1}\langle \asymp \rangle$, together with the rule $\langle D \coprod \bigcirc\rangle = \delta\cdot\langle D\rangle$ (applicable also to link diagrams), allow us to express the symbol $\langle T\rangle$ as a formal linear combination $\langle T\rangle = f(T)\langle 0\rangle + g(T)\langle\infty\rangle$, where $\langle 0\rangle, \langle\infty\rangle$ are to be regarded as primitive objects, and where the coefficients $f(T), g(T)$ are in the ring $\mathbb{Z}[a,a^{-1}]$. We define the *bracket vector* of $T$ to be the ordered pair $(f(T), g(T))$, and denote it by $br(T)$. For example, $br(1) = (a, a^{-1})$. Where appropriate, we shall consider $br(T)$ as the column vector

$$\begin{bmatrix} f(T) \\ g(T) \end{bmatrix}.$$

The identities of the next proposition can be confirmed merely by verifying that they hold for the generators $0, \infty$, and then applying linearity.

**Proposition 2.2.**

(i) $\begin{bmatrix} \langle T^{\mathrm{N}}\rangle \\ \langle T^{\mathrm{D}}\rangle \end{bmatrix} = \begin{bmatrix} \delta & 1 \\ 1 & \delta \end{bmatrix} br(T).$

(ii) $br(T+U) = \begin{bmatrix} f(U) & 0 \\ g(U) & f(U)+\delta g(U) \end{bmatrix} br(T)$

and

$$br\ (T * U) = \begin{bmatrix} \delta f(U) + g(U) & f(U) \\ 0 & g(U) \end{bmatrix} br(T).$$

Returning to $H(T,U)$, we observe that if we take $(T,U) = (0,0)$, we obtain the 2-strand parallel of the standard diagram of the Hopf link. The bracket polynomial of this diagram, namely $-a^{-14} - a^{-6} - 2a^{-2} - 2a^2 - a^6 - a^{14}$, is not hard to compute by hand. The choice $(T,U) = (0,\infty)$ (or $(T,U) = (\infty, 0)$) yields a diagram with writhe 0 of the unlink of 3 components, and $(T,U) = (\infty,\infty)$ gives a diagram with writhe 0 of the unlink of 2 components. Therefore, the bracket polynomials of $H(0,\infty), H(\infty,\infty)$ are $\delta^2, \delta$, respectively. For convenience, let us define

$$h_{00} = \langle H(0,0)\rangle = -a^{-14} - a^{-6} - 2a^{-2} - 2a^2 - a^6 - a^{14},$$
$$h_{01} = h_{10} = \langle H(0,\infty)\rangle = \delta^2,$$
$$h_{11} = \langle H(\infty,\infty)\rangle = \delta$$

and let $\mathscr{H}$ denote the matrix

$$\begin{bmatrix} h_{00} & h_{01} \\ h_{10} & h_{11} \end{bmatrix}.$$

From the bracket expansion formula, we immediately have the following alternative formula for the bracket polynomial of $H(T,U)$:

**Proposition 2.1′.** $\langle H(T,U)\rangle = h_{00}f(T)f(U) + h_{01}(f(T)g(U) + g(T)f(U)) + h_{11}g(T)g(U)$ *or, in matrix notation*

$$\langle H(T,U)\rangle = br(T)^t \cdot \mathscr{H} \cdot br(U).$$

The equivalence of the two formulae for $\langle H(T,U)\rangle$ may be demonstrated by means of Proposition 2.2.

We turn now to geometric properties of the link $H(T,U)$. Recall from [9] that a link $L$ in a solid torus $V$ is said to be *geometrically essential in* $V$ if each cross-sectional disk of $V$ meets $L$. The sublinks $T^{\mathrm{N}}, U^{\mathrm{N}}$ of $H(T,U)$ lie in solid tori $V_T, V_U$, respectively, whose cores form a Hopf link. If $T^{\mathrm{N}}, U^{\mathrm{N}}$ are geometrically essential in their respective solid tori, then the boundaries of the solid tori are incompressible in $S^3 - H(T,U)$, and the cores of $V_T, V_U$ form a companion Hopf link of $H(T,U)$. On the other hand, if one of $T^{\mathrm{N}}, U^{\mathrm{N}}$ is not geometrically essential, then the link $H(T,U)$ is split by a 2-sphere separating $T^{\mathrm{N}}$ from $U^{\mathrm{N}}$. We are particularly interested in finding properties of the tangles $T,U$ which guarantee that: (i) $H(T,U)$ is non-split and (ii) $H(T,U)$ is prime.

In order to discuss tangles satisfactorily in geometric terms, it is necessary to consider a tangle as a pair $(B,T)$, where $B$ is a 3-ball and $T$ is a proper 1-submanifold of $B$ meeting the boundary of $B$ in four points. When viewing a diagram of a tangle, it is understood that $B$ is a Euclidean 3-ball whose boundary meets the projection plane in a "equatorial" circle circumscribing the tangle diagram, and that $T$ itself lies in the projection plane except for small vertical perturbations near crossings. A tangle $(B,T)$ is *trivial* or *rational* if it is homeomorphic to a pair $(B,T_0)$, where $T_0$ is the union of two parallel line segments in the projection plane, for example the zero tangle.

**Definition.** Let $(B,T)$ be a tangle, presented as a tangle diagram in the plane. A *separating disk for* $T$ is a properly embedded disk in $B$ that avoids $T$ and separates the endpoints of $T$ into two pairs. An *NS-separating disk for* $T$ is a separating disk for $T$ whose boundary is the great circle on $\partial B$ that lies in a north-south vertical plane.

For example, any rational tangle admits a separating disk, but the only rational tangle admitting an NS-separating disk is the tangle $\infty$: **)(**.

If $T$ is one of the substituent tangles in $H(T,U)$, then the following statements are equivalent: (i) $(B,T)$ admits an NS-separating disk; (ii) the numerator $T^{\mathrm{N}}$ lies in a 3-ball in $V_T$; and (iii) $T^{\mathrm{N}}$ fails to be geometrically essential in $V_T$.

**Definition.** A tangle $(B,T)$ *has a connected summand* if there exists a 2-sphere in $B$ which meets $T$ in two points and which bounds a 3-ball in $B$ whose intersection with $T$ is other than an unknotted arc.

Informally, a connected summand of a tangle $T$ is a non-trivial link spliced into an arc of $T$. A connected summand of $T$ will persist as a connected summand of $T^{\mathrm{N}}$; therefore, if we wish $H(T,U)$ to be prime, we must use substituent tangles that are free of connected summands.

If $(B,T)$ is non-trivial and has a separating disk $\Delta$, then the union of $\Delta$ with one of the components of $\partial B - \partial\Delta$ will be a 2-sphere exhibiting a connected summand of $(B,T)$. For this reason, the only tangles with separating disks that we shall use in the construction of links of form $H(T,U)$ are rational tangles.

A separate observation is that since there exists a cross-sectional disk of $V_T$ meeting $T^{\mathrm{N}}$ transversely in two points, every cross-sectional disk of $V_T$ which is transverse to $T^{\mathrm{N}}$ must meet $T^{\mathrm{N}}$ in an even number of points.

The next two propositions use elementary general position arguments to establish sufficient conditions for $H(T,U)$ to be a prime link.

**Proposition 2.3.** *Suppose that neither of $T$, $U$ is separated by a* 2-*sphere in its ambient* 3-*ball, and that neither of $T$, $U$ admits an NS-separating disk. Then the link $H(T,U)$ is non-split.*

**Proof.** From the second part of the hypothesis, each of $T^{\mathrm{N}}, U^{\mathrm{N}}$ is geometrically essential in its solid torus. Let $F$ be a 2-sphere in $S^3 - H(T,U)$, and suppose that $F$ separates the link $H(T,U)$. $F$ cannot separate the two companion tori, as the Hopf link is not split; therefore, $F$ must meet a companion torus, say $\partial V_T$, and we may assume that $F \cap (\partial V_T \cup \partial V_U)$ is the union of finitely many disjoint simple closed curves. We may also assume that the number of components of $F \cap (\partial V_T \cup \partial V_U)$ cannot be reduced by an isotopy of $F$. Let $C$ be a simple closed curve of intersection that is innermost on $F$, say $C \subset F \cap \partial V_T$. If $C$ is homotopically non-trivial on $\partial V_T$, then the innermost disk on $F$ bounded by $C$ is a cross-sectional disk for $V_T$ avoiding $T^{\mathrm{N}}$, contradicting the fact that $T^{\mathrm{N}}$ is geometrically essential. On the other hand, if $C$ is homotopically trivial on $\partial V_T$, then the fact that $C$ cannot be removed by an isotopy of $F$ implies that the 2-sphere formed by the disk on $\partial V_T$ bounded by $C$ and the innermost disk on $F$ bounded by $C$ bounds a 3-ball in $V_T$ meeting $T^{\mathrm{N}}$. If this 3-ball contains the two arcs forming the numerator closure of $T$, then again we would have an NS-separating disk for $T$; otherwise we would have a 2-sphere in the ambient 3-ball of $T$ separating $T$. $\square$

**Proposition 2.4.** *Suppose that $T,U$ meet the conditions of Proposition* 2.3, *and that neither of $T$ nor $U$ has a connected summand. Then the link $H(T,U)$ is prime.*

**Proof.** From the first part of the hypothesis, the tori $\partial V_T, \partial V_U$ are incompressible in $S^3 - H(T,U)$; also, by Proposition 2.2, $H(T,U)$ is non-split, whence $S^3 - H(T,U)$ is irreducible. Let $F$ be a 2-sphere in $S^3$ meeting $H(T,U)$ transversely in two points. These points must lie on the same component of $H(T,U)$; hence they lie in the same companion solid torus, say without loss of generality $V_T$. Since $\partial V_T$ is incompressible and $S^3 - H(T,U)$ is irreducible, we may isotope $F$ so as to remove all simple closed curves of $F \cap \partial V_T$ which are homotopically trivial in $F - (F \cap H(T,U))$. We are then left with a finite number of parallel simple closed curves on $F$ that separate the two points of

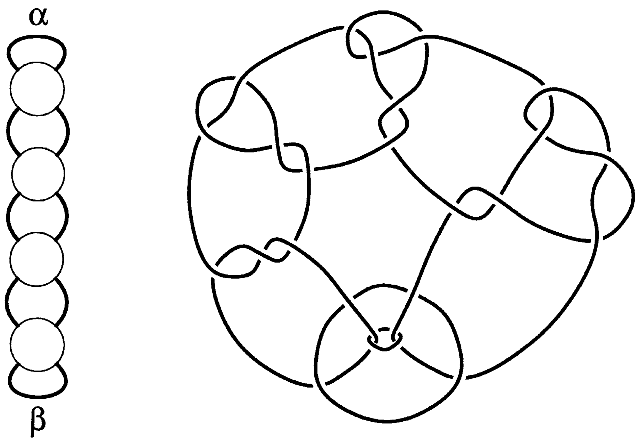



Fig. 4. (i) & (ii).

$(F \cap H(T, U))$. If the number of such curves is zero, then $F$ lies in $V_T$; since by hypothesis $T$ has no connected summands, $F$ bounds a 3-ball in $V_T$ meeting the link $H(T, U)$ in an unknotted arc, and we are done. Otherwise, the two curves closest to the respective points of $(F \cap H(T, U))$ bound disjoint cross-sectional disks of $V_T$ meeting $H(T, U)$ in a single point; this is impossible, as each such cross-sectional disk meets $H(T, U)$ in an even number of points. $\square$

The hypotheses of Propositions 2.3 and 2.4 can be met even if the sublinks $T^{\mathrm{N}}, U^{\mathrm{N}}$ are trivial or composite. For example, the link $H(T, U)$ illustrated in Fig. 4(ii) is prime, even though $T^{\mathrm{N}}$ is a connected sum of four links and $U^{\mathrm{N}}$ is the unlink of two components.

The final proposition of this section deals with the issue of connected sums, and will be useful for constructing prime links with prescribed polynomials.

**Definition.** A tangle $T$ is *primary* if it meets the hypotheses of Propositions 2.3 and 2.4, namely if $T$ is not separated by a 2-sphere in its ambient 3-ball, $T$ does not admit an NS-separating disk, and $T$ has no connected summand.

**Proposition 2.5.** *Let $L$ be a non-split link in $S^3$. Then there exists a primary tangle $T$ such that $T^{\mathrm{N}} = L$.*

**Proof.** If $L$ is the unknot, we may take $T = 1$. Otherwise, from the hypothesis, $L$ is either prime or a connected sum of prime links. Let us suppose first that $L$ is prime. We take any diagram of $L$, and then choose two edges $\alpha, \beta$ of the projection of $L$ sharing a common region and corresponding to distinct Wirtinger generators of the link group $\pi_1(S^3 - L)$. Cutting out interior segments of these edges and choosing a suitable coordinate system yields a diagram of a tangle $T$ with numerator equal to $L$. Since we are assuming for the moment that $L$ is prime, all conditions for $T$ being primary are obviously met except possibly for the condition regarding the absence of an NS-separating disk for $T$. But such a disk would extend to a 2-sphere meeting $L$ in a point of $\alpha$ and a point of $\beta$, impossible as $\alpha, \beta$ correspond to distinct Wirtinger generators.

The proof for the case where $L$ is composite is very similar, except that we need to choose the arcs $\alpha, \beta$ carefully in order to fulfill the condition that $T$ should have no connected summand. Specifically, we first choose a diagram of $L$ where all the connected summands are arranged in a chain, in the manner of Fig. 4(i) (to achieve this configuration it might be necessary to "feed" one connected summand through another, as explained in [2]). We then take $\alpha, \beta$ to be arcs at the two extremities of the chain, as indicated. This action will "purge" all connected summands from $T$. □

## 3. Infinite sequences of links with common bracket polynomial

In this section we describe a general way of generating infinite sequences of links with common bracket and Jones polynomials.

**Definition.** Given a tangle $T$, $T^{\omega}$ denotes the tangle $(T+2)\cdot 1\cdot 2$, and $T^{\bar{\omega}}$ denotes the tangle $(T-2)\cdot(-1)\cdot(-2)$ (Fig. 5).

Using the operation $*$ introduced in Section 1, we may also write $T^{\omega}=((T+2)*1))+2$, and $T^{\bar{\omega}}=((T-2)*(-1))-2$.

We may consider $\omega$ as a self-homeomorphism of the (3-ball, tangle) pair $(B,T)$, mapping its boundary by a self-homeomorphism of $(\partial B, \partial B\cap T)$ that interchanges the SW and SE endpoints of $T$.

It may easily be verified that $T^{\omega\bar{\omega}}$ is equivalent to $T$ via an isotopy fixing the endpoints of $T$. This corresponds to the fact that the homeomorphisms of $(\partial B, \partial B\cap T)$ induced by $\omega$, $\bar{\omega}$ represent inverse elements of the mapping class group of $(\partial B, \partial B\cap T)$.

We note the following elementary properties of the operator $\omega$:

(i) any given orientation of $T^{\mathrm{N}}$ extends to an orientation of $(T^{\omega})^{\mathrm{N}}$, whence signs of crossings within $T^{\mathrm{N}}$ are preserved when transforming to $(T^{\omega})^{\mathrm{N}}$;

(ii) the sum of the signs of the five additional crossings in $(T^{\omega})^{\mathrm{N}}$ is always $+1$.

In (ii) we are taking into account the fact that an orientation of the numerator of a tangle $T$ forces one of the NW, NE ends of $T$ to be directed inwards, and the other outwards.

Of course, the operator $\bar{\omega}$ enjoys the same properties, except that the sum of the signs of the five additional crossings is always $-1$.

We now determine the effect that the operations $\omega, \bar{\omega}$ have on the bracket vector of a tangle. From Proposition 2.2(ii) we have

$$br(T+1)=M_{+}\cdot br(T),\quad br(T*1)=M_{*}\cdot br(T),$$

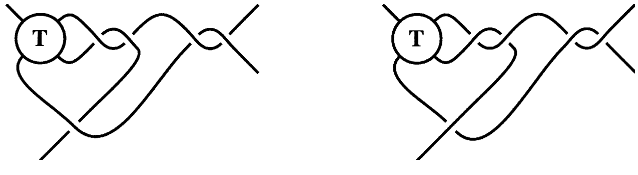



Fig. 5. The tangles $T^{\omega}$, $T^{\bar{\omega}}$.

where

$$M_+ = \begin{bmatrix} a & 0 \\ a^{-1} & -a^{-3} \end{bmatrix}, \qquad M_* = \begin{bmatrix} -a^3 & a \\ 0 & a^{-1} \end{bmatrix}.$$

In the present context it is natural to introduce the $2 \times 2$ matrix

$$\Omega = M_+^2 M_* M_+^2 = \begin{bmatrix} -a^{-1} + a^3 - a^7 & a^{-3} \\ -a^{-11} + 2a^{-7} - 2a^{-3} + 2a - a^5 & a^{-13} - a^{-9} + a^{-5} \end{bmatrix}.$$

We then have

**Proposition 3.1.**

$$br(T^{\omega}) = \Omega \cdot br(T),$$

$$br(T^{\bar{\omega}}) = \Omega^{-1} \cdot br(T).$$

**Definition.** Given tangles $T, U$, $H(T,U)^{\omega}$ denotes the diagram $H(T^{\omega}, U^{\bar{\omega}})$.

**Theorem 3.2.** *Let $T, U$ be any tangles. Then the bracket polynomials of $H(T,U)$, $H(T,U)^{\omega}$ are equal.*

**Proof.** This follows from Proposition 3.1 and the easily verifiable identity

$$\Omega^t \mathscr{H} \Omega^{-1} = \mathscr{H},$$

where $\mathscr{H}$ is the matrix in the formula $\langle H(T,U) \rangle = br(T)^t \cdot \mathscr{H} \cdot br(U)$ of Proposition 2.1′. □

The writhe of $H(T,U)^{\omega}$ might differ from that of $H(T,U)$, as application of $\omega$ interchanges two tangle ends, and can therefore affect the signs of the eight "clasp" crossings in the pattern $H(T,U)$ where the two numerators meet. However, a double application of $\omega$ preserves tangle ends, and in view of properties (i) and (ii) of $\omega$ stated above we have the following additional result:

**Theorem 3.2′.** *Let $T, U$ be any tangles. Then the Jones polynomials of $H(T,U)$, $H(T,U)^{\omega^2}$ are equal, assuming that the transformed tangles are oriented in a manner consistent with the orientations of the original tangles.*

By iterating the transformation $H(T,U) \to H(T,U)^{\omega}$, from given tangles $T, U$ we can construct an infinite sequence of links, such that alternate links in the sequence have the same Jones polynomial (all diagrams in the sequence have the same bracket polynomial).

Let $T$ be a primary tangle. Then either $T$ is a rational tangle, or $T$ has no separating disk. In the former case, the sequence $T, T^{\omega}, T^{\omega^2}, \ldots$ contains at most one instance of a tangle admitting an NS-separating disk, whereas in the latter case no tangle in the sequence can admit such a disk, as the existence of a separating disk is a topological property.

We are now ready to state and prove our main results.

**Theorem 3.3.** *Let $L$ be any non-split link with $k \geqslant 1$ components. Let $V_L$ denote the Jones polynomial of $L$, and let $u$ denote the Jones polynomial of the 2-component unlink, i.e. $u = -t^{-1/2} - t^{1/2}$. Then there are infinitely many inequivalent prime $(k+1)$-component links with Jones polynomial equal to $uV_L$.*

**Proof.** First we consider the case where $L$ is distinct from the unknot. By Proposition 2.5 there exists a primary tangle $T$ with numerator $L$. Let $U$ be the tangle $\infty + 2$, i.e. the tangle . We define a sequence of $(k+1)$-component links $\Lambda_i$ $(i \geqslant 0)$ inductively as follows: $\Lambda_0 = H(T, U)$, $\Lambda_i = \Lambda_{i-1}^{\omega^2}$ $(i \geqslant 1)$. As $\Lambda_0$ is a split union of $L$ with the unknot, the Jones polynomial of $\Lambda_0$ is $uV_L$. Therefore, by Theorem 3.2′, all $\Lambda_i$ have the desired Jones polynomial. It therefore remains to be shown that there are infinitely many distinct prime links amongst the $\Lambda_i$.

From the discussion immediately before this theorem, at most one tangle in the sequence $T, T^{\omega^2}, T^{\omega^4}, \dots$ fails to be primary. The tangle $U = \infty + 2$ is not primary; however, repeated application of $\bar{\omega}^2$ to $\infty + 2$ yields the sequence of rational tangles

$$-(5 \cdot 1 \cdot 2), -(5 \cdot 1 \cdot 4 \cdot 1 \cdot 4 \cdot 1 \cdot 2), -(5 \cdot 1 \cdot 4 \cdot 1 \cdot 4 \cdot 1 \cdot 4 \cdot 1 \cdot 4 \cdot 1 \cdot 2) \dots .$$

Since these rational tangles are all primary, by Proposition 2.3 at most one of the links $\Lambda_i$ $(i \geqslant 1)$ can fail to be prime; since the numerators of these tangles are pairwise distinct 2-bridged knots, there must be infinitely many link types amongst the $\Lambda_i$.

If $L$ is the unknot, we choose $T = \infty - 2$, $U = \infty + 2$, and construct the sequence $\Lambda_i$ as before. The situation here is simpler, as $\Lambda_i = H(T_i, -T_i)$, where $T_i = 5 \cdot (1 \cdot 4)^{2(i-1)} \cdot 1 \cdot 2$. Clearly, the tangles $T_i$ are all primary for $i \geqslant 1$, and the numerators of the $T_i$ are pairwise distinct 2-bridged knots. Therefore the conclusion holds in this case also. □

For split links a slightly stronger statement is possible, in view of the special nature of the Jones polynomial of a split link.

**Theorem 3.3′.** *Let $L$ be an arbitrary link of $k$-components; let us suppose that $L$ is a split union of links $L_1, L_2, \dots, L_m$ $(m \geqslant 1)$, where the splitting is maximal in that each $L_i$ is non-split. Then for each $i \geqslant -m + 2$ there are infinitely many $(k+i)$-component prime links with Jones polynomial equal to $u^i V_L$.*

**Proof.** The connected sum $L_1 \# L_2 \# \cdots \# L_m$ is a non-split $(k - m + 1)$-component link with Jones polynomial $u^{-m+1} V_L$. Apply Theorem 3.3 repeatedly to this connected sum. □

**Corollary 3.3.1.** *For each $k \geqslant 2$ there are infinitely many prime $k$-component links having the same Jones polynomial as the $k$-component unlink.*

In the next section we describe examples of this construction.

## 4. Sequences of links with trivial polynomials

### *4.1. The family of 2-component links $LL_2(n)$ (Fig. 6)*

Our first example is the sequence generated by the pair $T = \infty - 2$, $U = -T = \infty + 2$. This sequence featured in the part of the proof of Theorem 3.3 concerned with the unknot. We use the subscript 2

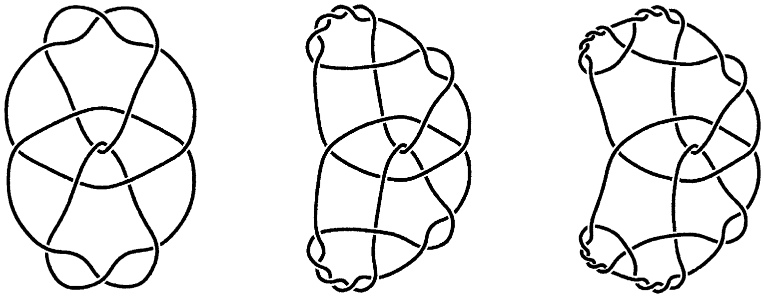


Fig. 6. The links $LL_2(1)$, $LL_2(2)$, $LL_2(3)$.

in its identifier as it was the second such sequence to be discovered. $H(\infty-2,\infty+2)$ is a diagram of the unlink of two components with writhe 0; therefore, repeated applications of the operator $\omega$ to the tangle $T$ yields a sequence of rational tangles $T_0=\infty-2$, $T_1=3$, $T_2=5\cdot 1\cdot 2$, $T_3=5\cdot 1\cdot 4\cdot 1\cdot 2,\ldots,$ such that $\langle H(T_n,-T_n)\rangle=\delta$ for all $n\geqslant 0$.

**Definition.** $LL_2(n)=H(T_n,-T_n)$, where the tangle $T_n$ is the result of $n$ applications of the operator $\omega$ to the tangle $\infty-2$.

It can be verified that the writhe of $H(T_n,-T_n)$ is zero for even $n$, whereas for odd $n$ the writhe is equal to $\pm 8$, the sign depending on choice of string orientations.

Let $V_L(t)$ denote the Jones polynomial of a link $L$, and let $u$ denote the Jones polynomial of the 2-component unlink, i.e. $u=-t^{-1/2}-t^{1/2}$. From the discussion of the previous paragraph, we may assert:

**Theorem 4.1.** *$V_{LL_2(n)}(t)=u$ for even $n$, and $V_{LL_2(n)}(t)=t^{\pm 6}u$ for odd $n$, where the sign depends on choice of string orientations.*

For even $n$ the link $LL_2(n)$ has the added distinction of having zero Alexander polynomial, on account of being a boundary link: it is easily checked by means of Seifert's algorithm that the sublinks $T^{\mathrm{N}}, U^{\mathrm{N}}$ bound disjoint Seifert surfaces. Therefore the sequence $LL_2(n)$ $(n=2,4,6,\ldots)$ is an infinite sequence of pairwise distinct prime links indistinguishable from the unlink by both Jones and Alexander polynomials. However, although we have not proved this, it appears that the Homfly polynomials of these links are all non-trivial.

### *4.2. The 2-parameter family of 2-component links $LL_1(m,n)$ (Fig. 8)*

Let us define a 2-parameter family of tangles $T_{m,n}=n\cdot 1\cdot(1\cdot 1/2)^m\cdot(-1)$, where $m\geqslant 0$ and $n$ is any integer.

**Proposition 4.2.** *The numerator of $T_{m,n}$ is the unknot.*

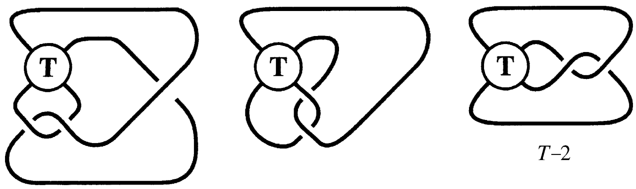



Fig. 7.

**Proof.** In the case $m=0$, a single Reidemeister move of type II transforms the numerator of $T_{m,n}$ to the denominator of the tangle $n$, so $(T_{0,n})^{\mathrm{N}}$ is indeed the unknot.

Now let us consider the case $m>0$. If $T$ is any tangle, inspection of Fig. 7 shows that the numerator of $T\cdot(1/2)\cdot(-1)$ is equivalent to that of $T-2$. Therefore the numerators of $T_{m,n}$, $T_{m-1,n}$ are equivalent, and the conclusion follows by induction on $m$. □

A single application of $\omega$ to $T_{m,n}$ and $\bar{\omega}$ to $\infty+2$ leads to the following family.

**Definition.** $LL_1(m,n)=H(T,-3)$, where $T=(T_{m,n})^{\omega}=n\cdot 1\cdot(1\cdot 1/2)^m\cdot 1\cdot 1\cdot 2$.

It is easily checked that the transformation from $H(T_{m,n},\infty+2)$ to $H(T^{\omega}_{m,n},-3)$ does not alter writhe when $m\geqslant 1$, or when $m=0$ and $n$ is even. However, when $m=0$ and $n$ is odd, the contribution to the writhe from the eight clasp crossings changes from 0 to $\pm 8$. Therefore, we may assert:

**Theorem 4.3.**

$$V_{LL_1(m,n)}(t)=\begin{cases} u & (m\geqslant 1),\\ u & (m=0 \quad \text{and } n \text{ even}),\\ t^{\pm 6}u & (m=0 \quad \text{and } n \text{ odd}).\end{cases}$$

Clearly the transformation $\omega$ can be applied to $LL_1(m,n)$ to generate a 3-parameter infinite family of 2-component links with trivial Jones polynomial.

### *4.3. The family of 3-component links $LLL(n)$ (Fig. 9)*

Let $T_n$ be the tangle $2\cdot(1/2\cdot 1)^{n-1}\cdot -1$ $(n\geqslant 1)$, and let $T_0$ be the zero tangle. An almost identical argument to that of Proposition 4.2 shows that the numerator of $T_n$ is the unlink of two components, for all $n\geqslant 0$. A single application of $\omega$ yields the following family:

**Definition.** $LLL(n)=H(T,-3)$ $(n\geqslant 0)$, where $T=(T_n)^{\omega}=2\cdot(1/2\cdot 1)^n\cdot 1\cdot 2$.

The following is easily verified.

**Theorem 4.4.** *$V_{LLL(n)}(t)=u^2$ for $n\geqslant 1$, and $V_{LLL(0)}(t)=u^2$ if the components of $T^{\mathrm{N}}$ are oriented so that the linking number between $T^{\mathrm{N}}$ and $U^{\mathrm{N}}$ is zero.*

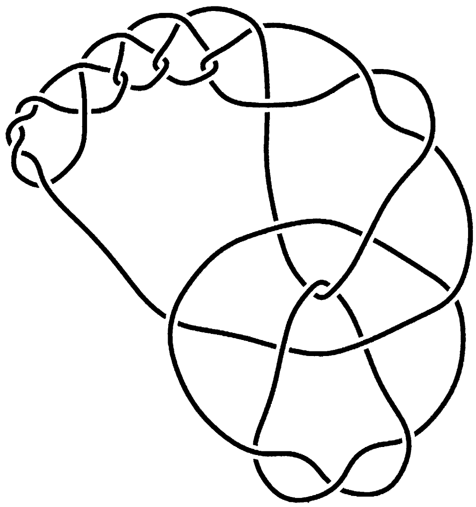


Fig. 8. $LL_1(3,4)$.

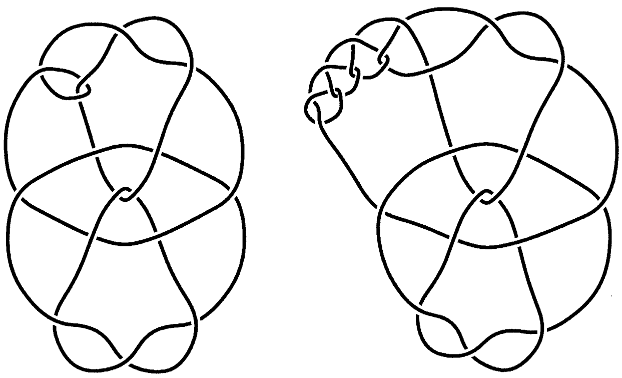


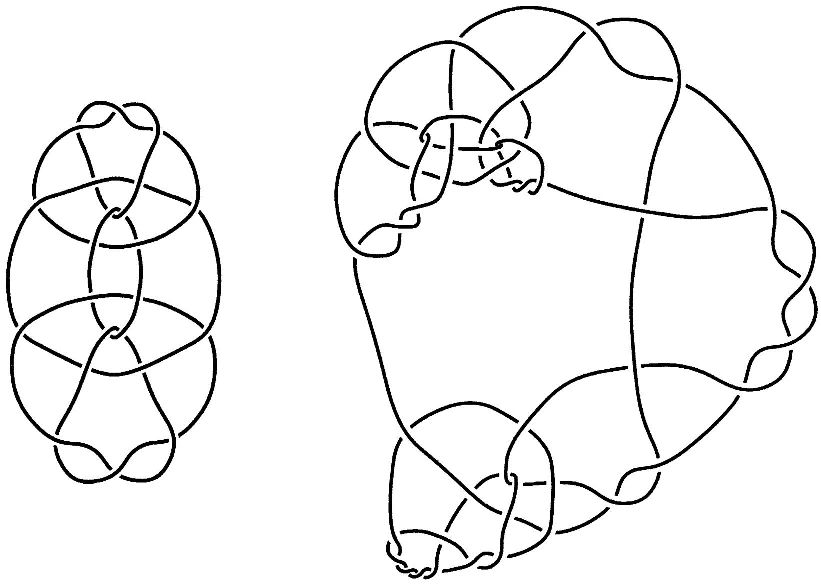
Fig. 9. $LLL(0)$, $LLL(3)$.

## 5. Some remarks on the families

1. The links $LL_1(0,-2)$, $LL_1(0,-1)$, $LL_1(0,0)$, $LL_1(0,1)$ and $LLL(0)$ were originally discovered by the third author during the course of a computer enumeration of links.

2. The links with trivial polynomial described here are all effectively classifiable, as their sublinks are all alternating, and the way in which they sit inside the regular neighbourhood of the companion Hopf link is evident. It is perhaps ironic that the Jones polynomial, whose discovery played a fundamental rôle in the proof of the Tait conjectures [4,6,7], fails completely to distinguish these links directly. There are a few isolated duplications within the families, for instance $LL_1(m+1,-1)= LL_1(m,-2)$ and $LL_1(0,-1)=LL_2(0)$.

3. In cases where the individual numerators $T^N, U^N$ are presented in reduced alternating form and the linking number between the numerators is $\pm 4$, the diagram $H(T,U)$ of Fig. 1 must have



(i) (ii)

Fig. 10. Prime 4- and 5-component links with trivial Jones polynomial.

minimal crossing-number, as a linking number of $n$ cannot be realized with fewer than $2|n|$ crossings with strands in both sublinks. It is possible that a more refined argument could be used to show that $H(T, U)$ has minimal crossing-number for reduced alternating $T, U$, without this assumption regarding linking number.

4. One can try to generalize the $H(T, U)$ construction. For example, an analogue of Theorem 3.2 holds for the 2-clasp pattern of Fig. 10(i). Therefore the argument of Theorem 3.3 shows that the 4-component link illustrated in Fig. 10(i) has Jones polynomial equal to that of the 4-component unlink, if one orients the middle two components so as to make all linking numbers zero. Fig. 10(ii) illustrates a prime 5-component link with trivial Jones polynomial, obtained from the link of Fig. 10(i) by the method of Theorem 3.3.

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