SECTION 9.1 STRICTLY DETERMINED GAMES

We open the discussion of Game Theory first by noting we want only to look at aspects of what are called "Zero Sum Games". In these games, the Zero Sum comes from this notion: whatever money one player wins is exactly the amount of money the other player loses. Simply put, it is an exchange of money from the losing player to the winning player, there are no third parties taking a piece of the pie.

For simplicity, we will think of our 2 players as R(for Rows) and C(for Columns), where each players possible outcomes from the Game are represented in a matrix. For example, we have the below Payoff Matrix:

$$\begin{array}{cc} & C \\ R & \begin{bmatrix} 2 & -1 & 3 \\ -3 & -2 & 4 \end{bmatrix} \end{array}$$

Player R looks at the Rows to see their possible payouts. If R chooses to play the option in Row 1, for example, the payoff will be a gain of \$2, a loss of \$1, or a gain of \$3. If R plays Row 2, payoffs possible are a loss of \$3, a loss of \$2, or a gain of \$4.

On the other hand, Player C looks at the Columns to see their payoffs, but must take the OPPOSITE of each number to give their gain/loss. For example, in Column 1, C sees a 2 and a -3. But this means a LOSS of \$2 or a GAIN of \$3. Remember, we said that R would possibly see a gain of \$2 in choosing to play Row 1, and that \$2 must come from player C, so a loss for C.

Let us suppose R sees the 4 in Row 2 and decides to play Row 2 over and over, hoping to hit that \$4 gain, while C notices the -3(so a gain of \$3 for C) in Column 1 and decides to play Column 1 over and over. As things stand, C will win \$3 from R each play unless R changes things up. Which R logically would do, switching to Row 1, since R would win \$2 from C if C keeps playing Column 1. However, now C will realize that repeated plays of Row 1 are occurring, so should choose to move from Column 1 to Column 2, where C now wins \$1 each play.

We are now seeing R play Row 1 repeatedly, and C play Column 2 repeatedly. Does either of them improve their situation by making another change of strategy while their opponent does not change? If you think about it, the answer is "No", neither of them will switch to another option because to do so will hurt their current gain/loss position.

When such a situation occurs, it is called a Strictly Determined Game. The specific gain/loss figure their respective choice targets is called the Value, v, of the game. Here, we have that v = -1, a loss of \$1 for R and so a gain of \$1 for C every time the game is played.

The position in the Payoff Matrix is referred to as a Saddle Point, a value which is both the Minimum of its Row and the Maximum of its Column. Verify in our Payoff Matrix that this is the case in Row 1, Column 2, that the -1 is a Saddle Point in this sense.

Now, we discuss how to identify whether a Payoff Matrix has a Saddle Point or not, and so whether the Game is Strictly Determined or not.

One fairly simple way to do this is to look across each Row and Circle the lowest(minimum) number, and then look at each Column and Box the highest(maximum) number in each.

If you have both Circled and Boxed a specific number, it must have been both a Minimum of its Row as well as a Maximum of its Column, the exact description we just said makes it a Saddle Point.

Try it on our Payoff Matrix:

$$\begin{array}{cc} & C \\ R & \begin{bmatrix} 2 & -1 & 3 \\ -3 & -2 & 4 \end{bmatrix} \end{array}$$

Now try it on the following Payoff Matrices, and decide whether each is Strictly Determined or not. And if it is Strictly Determined, what is its Value, v?

i) $\begin{array}{cc} & C \\ R & \begin{bmatrix} -2 & -1 & 3 \\ 3 & -2 & 0 \end{bmatrix} \end{array}$

ii) $\begin{array}{cc} & C \\ R & \begin{bmatrix} 2 & 8 \\ 5 & 6 \end{bmatrix} \end{array}$

iii) $\begin{array}{cc} & C \\ R & \begin{bmatrix} 3 & -1 & 4 \\ -2 & -2 & 2 \\ 0 & -3 & 5 \end{bmatrix} \end{array}$

APPLICATION: Rick and Chuck are playing a game. Rick has cards with the numbers 2, 5, and 6, while Chuck has cards with the numbers 3, 4, and 9. They must each show a card of their own choosing. If they show cards where one is Even and one is Odd, Rick wins the difference of card values, but if they show cards that are both Even or both Odd, Chuck wins the value of the larger card showing.

a) Construct a labeled Payoff Matrix to reflect the gains/losses each player will incur
b) Determine whether the Payoff Matrix is Strictly Determined. If so, what is the Value?

APPLICATION: Rhonda and Chrissy are competing drive-share operators. Both of them use $\ddot{O}ver$ and *Ryze* phone apps to locate people seeking rides around town. When both are using $\ddot{O}ver$, Rhonda takes \$100 in commissions from Chrissy, while if both use *Ryze*, Chrissy takes \$250 from Rhonda. If they are using different apps, Rhonda using $\ddot{O}ver$ takes \$150 from Chrissy, while if Chrissy uses $\ddot{O}ver$, she takes \$75 from Rhonda.

a) Construct a labeled Payoff Matrix to reflect the gains/losses each player will incur
b) Determine whether the Payoff Matrix is Strictly Determined. If so, what is the Value?

SECTION 9.2 MIXED STRATEGIES

The Strictly Determined Games in 9.1 leave no doubt how the games outcomes will occur after repeated plays of the game force the players into the Saddle Point. So, what might the Value, v, of a game be when it is not Strictly Determined? This will depend on how often each player chooses to play their respective Row/Column options.

We will continue using matrices to find our answer. Thus, let us put the probabilities of R playing each Row into a Row Matrix, $R = \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix}$, and the probabilities of C playing each Column into a Column Matrix, $C = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$. Of course, how many elements we have in each of the matrices R and C depends on how many options each has in the Payoff Matrix, which we will call A.

We now can use a very simple Matrix Multiplication, $R \cdot A \cdot C$, the result of which is a 1x1 matrix containing the Value, v, of this particular situation. Understand that v is most likely not an element in the Payoff matrix, but a number that would be the "average" amount won/lost by each player over many plays of the game using the assigned probabilities. If v is positive, R wins on average, and if v is negative, C wins on average.

Given $A = \begin{bmatrix} -2 & 3 \\ 1 & -4 \end{bmatrix}$, first analyze it to verify that it is NOT Strictly Determined.

i) Let us assign probabilities for each player. Start with R playing Row 1 70% of the time, and C playing Column 1 40% of the time. Construct both matrices R and C, and the perform the calculation $R \cdot A \cdot C$ What do you get for *v*? Which player is winning on an average play right now?

Since we see a positive value of *v*, R is currently winning, and so we expect C might alter strategy.

ii) Keeping R's probabilities the same, change C to playing Column 1 20% of the time instead of 40%. What is the new value of *v*? Did C improve their situation?

iii) Because that change in strategy did not work for C, perhaps C flips probabilities around, now deciding to play Column 1 80% of the time. What is the new value of *v*? Did C improve their situation?

iv) Now that C seems to be winning on average, they might be content to continue their current % of plays on Columns 1 & 2. But now R might make a change, right? So, select some new values for R and see if the new choices improve the value of *v* in R's favor. List each row matrix R and the *v* it gives.

Probability plays are chosen by each side of the game, but the players will then attempt to improve their situation by adjusting their probabilities. If it improves the value, *v*, of the game in their favor, perhaps they hold steady, but their opponent is likely adjusting their play. Thus, what happens is both sides will continually try improving their outcome until perhaps we might find that neither can make any moves that will improve their situation.

To explore this, let us go back to the last probabilities for C, which was 80% play of Column 1. If you push R's plays as far as possible towards Row 2, in fact all the way to 100%, R does push *v* to a value of 0. It seems this is the best R can force to happen for now. But, let us have C make one more adjustment, to 70% in Column 1. So, we have now these matrices and value of *v*:

$$R = \begin{bmatrix} r_1 & r_2 \end{bmatrix} \quad A = \begin{bmatrix} -2 & 3 \\ 1 & -4 \end{bmatrix} \quad C = \begin{bmatrix} .7 \\ .3 \end{bmatrix} \qquad v = \begin{bmatrix} -0.5 \end{bmatrix}$$

You should have had $R = \begin{bmatrix} 0 & 1 \end{bmatrix}$ as the last used plays by R. Try changing R's probabilities to any valid choice you'd like. Does it improve R's position? Try again. And again. And again.

It turns out there is nothing R can do anymore, is there? C has found their best, Optimal play probabilities, and no matter what R does, the value of the game will be $0.50 in C's favor. We will explore this further in section 9.3 and find out using Linear Programming and Simplex how to determine what probabilities BOTH players can locate as their optimal strategies.

EXERCISE: We return to the game between Rick and Chuck, where the below matrix represents the payoffs for this game, labeled according to the cards each can choose to play.

$$\begin{array}{cc} & C \\ & \begin{array}{ccc} 3 & 4 & 9 \end{array} \\ \begin{array}{r} 2 \\ R\ 5 \\ 6 \end{array} & \begin{bmatrix} 1 & -4 & 7 \\ -5 & 1 & -9 \\ 3 & -6 & 3 \end{bmatrix} \end{array}$$

Choose starting probabilities for each side, calculate *v*, and determine who is currently winning on average. Then make adjustments to the probabilities for whichever player seems to be losing, to improve their average gain or loss in playing the game.

Are you able to find Optimal Strategy probabilities for either player? It is not too easy, is it?

SECTION 9.3 OPTIMAL STRATEGIES

Now we take a look at how do determine the Optimal Strategy for both players, R and C. A full description of the mathematical building of the following model was shown in lecture and can be read in your textbook. This worksheet presents the necessary steps for solving that mathematical problem.

Let us start with the Payoff Matrix below, which you should verify has no Saddle Point. This means that the Optimal Strategies for our players R and C will not be 100% for either of their possible plays. The first task will be to make all of the elements of the matrix positive numbers, and we prefer adding the smallest necessary Integer in accomplishing this. What would you add to each element of A below?

$A = \begin{bmatrix} -6 & 7 \\ 3 & -2 \end{bmatrix}$ Your altered Matrix: $\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$

Now that we have **ALL Positive** numbers in our matrix, we can proceed. It is far easier to solve this problem from the perspective of Player C, who needs to solve a Maximum Linear Programming problem, as opposed to R needing to solve a Minimum.

First, let us use C's generic probabilities: $c_1, c_2, \ldots, c_n$ for as many Columns in A as necessary. Multiply the Payoff matrix A times column C with the probabilities, where each row was shown to be $\leq v$ in the full description of the solution. For example, you might get $6c_1 + 2c_2 \leq v$ . Do so for our example.

Next, we will divide both sides of each Inequality by *v*, which is a positive value causing no changes to the inequality symbols. Continuing the example made above:

$6c_1 + 2c_2 \leq v$ becomes this: $6\frac{c_1}{v} + 2\frac{c_2}{v} \leq \frac{v}{v}$. Now, we introduce $z_i = \frac{c_i}{v}$, for each *i* from 1 to n. So,

$6\frac{c_1}{v} + 2\frac{c_2}{v} \leq \frac{v}{v}$ becomes $6z_1 + 2z_2 \leq 1$, which is a properly built Inequality constraint for use in Simplex.

Convert your Inequalities in $c_i\text{'s to } z_i\text{'s}$. Conveniently, our Objective function is to Maximize $z_1 + z_2$. Build the whole LP problem, which should resemble(but IS NOT) the one below:

$Maximize: \; z_1 + z_2$

$3z_1 + 8z_2 \leq 1$

$2z_1 + 4z_2 \leq 1$

$z_1 \geq 0, z_2 \geq 0$

Now, let's solve the LP problem, where it is best to use $y_i$'s as the Slack Variables because in a manner similar to building $z_i = \frac{c_i}{v}$, we also saw in the full solution that we build $y_i = \frac{r_i}{v}$, where of course the $r_i$'s are the probabilities Player R uses as their Optimal play probabilities. Fill in the below Initial Simplex Tableau:

$$\left[\begin{array}{ccccc|c} z_1 & z_2 & y_1 & y_2 & M & \\ \hline & & 1 & 0 & 0 & 1 \\ & & 0 & 1 & 0 & 1 \\ \hline & & 0 & 0 & 1 & 0 \end{array}\right]$$

Choose a Pivot Element, either column 1 or 2 is available, and continue pivoting until you reach the Optimal Solution to the problem. Write out your Final Tableau:

$$\left[\begin{array}{ccccc|c} z_1 & z_2 & y_1 & y_2 & M & \\ \hline & & & & 0 & \\ & & & & 0 & \\ \hline & & & & 1 & \end{array}\right]$$

We want to read off the values of the $z_i$'s in the "normal way" and the $y_i$'s in the "dual way". We also will need *v*. Let us remember we Maximized $z_1 + z_2$ which equals 1/*v*, and so *v* will always be found as the reciprocal of our optimal value of M in the final tableau.

Since $z_i = \frac{c_i}{v}$ for each *i*, we rearrange to get $c_i = v \cdot z_i$. Find, in fractions, each of our $c_i$'s

In similar fashion, we noted that $y_i = \frac{r_i}{v}$, so now $r_i = v \cdot y_i$. Find, in fractions, each of our $r_i$'s

One last item to address, which is that the *v* we found is for the adjusted matrix. To fix this, we reverse the very first step we made, where we chose to add an integer to each Payoff matrix element, and we now subtract that from the current *v* to get the correct *v* for the original Payoff Matrix.

You should have these values: $r_1 = 5/18$ $r_2 = 13/18$, $c_1 = 1/2$, $c_2 = 1/2$, $v = 1/2$

EXERCISES: find the Optimal plays for each player, R and C, in the below Payoff Matrices

i) $A = \begin{bmatrix} -4 & 1 \\ 3 & 2 \end{bmatrix}$  ii) $A = \begin{bmatrix} 2 & -4 \\ -1 & 6 \\ 3 & -5 \end{bmatrix}$  iii) $A = \begin{bmatrix} -2 & 1 & -1 \\ 0 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$

What do you notice about the Optimal Play probabilities in part (i)? What was the value, *v*? Go back and look at the original matrix, and check it for a saddle point.

APPLICATION: the rulers of two small countries, Rey and Caesar, need to decide each year which industry to put research funding into. When both rulers put funding into their energy sectors, Caesar's country will see a gain of $3,000,000 in trade between the two countries, while if both countries put funding into their technology sectors, Rey's country sees a gain of $4,000,000 in trade. If they put funding into different sectors, Rey's country gains $7,000,000 when he funds energy, while Caesar's country gains $5,000,000 when he funds the energy sector.

i) Build a Payoff matrix for this situation. Determine that it is NOT strictly determined, it has no saddle point.

ii) We would like to find the optimal strategies for the two leaders, but notice that using values like 3,000,000 or 7,000,000 are quite unwieldy, especially when we want to make all of the values in our Payoff Matrix positive. Adding 5,000,001 to each would result in truly imbalanced values for the purpose of solving a Simplex Tableau.

So, rebuild the Payoff matrix using numbers in Millions, for example using 3 for 3,000,000. We will just want to remember to think of our ultimate value *v* as also "in Millions".

iii) Use this new version of the Payoff Matrix to find the Optimal Strategies for each country's leader and decide which of their countries will have an expected gain in trade "on average" each year.